The Hilbert functions which force the Weak Lefschetz Property

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ABSTRACT. The purpose of this note is to characterize the finite Hilbert functions which force all of their artinian algebras to enjoy the Weak Lefschetz Property (WLP). Curiously, they turn out to be exactly those (characterized by Wiebe in $[Wi]$) whose Gotzmann ideals have the WLP.

This implies that, if a Gotzmann ideal has the WLP, then all algebras with the same Hilbert function (and hence lower Betti numbers) have the WLP as well. However, we will answer in the negative, even in the case of level algebras, the most natural question that one might ask after reading the previous sentence: If A is an artinian algebra enjoying the WLP, do all artinian algebras with the same Hilbert function as A and Betti numbers lower than those of A have the WLP as well?

Also, as a consequence of our result, we have another (simpler) proof of the fact that all codimension 2 algebras enjoy the WLP (this fact was first proven in $[HMNW]$, where it was shown that even the Strong Lefschetz Property holds).

Let $A = R/I$ be a standard graded artinian algebras, where R is a polynomial ring in r variables over a field k of characteristic zero, I is a homogeneous ideal of R, and the x_i's all have degree 1. We say that A enjoys the *Weak Lefschetz Property* (*WLP*) if, for a generic linear form $L \in R$ and for all indices $i \geq 0$, the multiplication map "$\cdot L$" between the k-vector spaces A_i and A_{i+1} has maximal rank (notice that, since A is artinian, $A_i = 0$

for $i \gg 0$, and therefore only a finite number of maps have to be considered).

The WLP is a fundamental and very natural property of artinian algebras, and has recently received much attention. For a broad overview and the main results achieved so far, see $[GHMS]$, $[HMNW]$ and $[MM]$ (along with their bibliographies), as well as three more recent works of M. Boij and the two authors of this note: $[Za]$, $[Mi]$, $[BZ]$.

In particular, one interesting problem that immediately arose was to study the structure of the Hilbert functions of the algebras having the WLP. In $[HMNW]$, the four authors characterized the Hilbert functions forcing *at least one* of their artinian algebras to have the Weak Lefschetz Property. See also Harima's paper $[Ha]$, where the Hilbert functions of Gorenstein algebras having the WLP are classified.

In this brief note, we will characterize the Hilbert functions which force *all* of their artinian algebras to enjoy the WLP. A curious fact is that these Hilbert functions will be exactly those (characterized by Wiebe in $[Wi]$, Theorem 4.2) whose Gotzmann ideals have the WLP.

As a consequence of our result, we have another (simpler) proof of the fact that all codimension 2 artinian algebras enjoy the WLP (this was first proven in $[HMNW]$, Proposition 4.4, where it was shown that even the Strong Lefschetz Property holds).

Our characterization, along with the result of Wiebe, immediately suggests the following question: If A is an artinian algebra enjoying the WLP, do all artinian algebras with the same Hilbert function as A and Betti numbers lower than those of A have the WLP as well?

We will prove that this is not the case, by exhibiting examples of codimension 3 level algebras (which are artinian reductions of algebras of reduced sets of points of the projective space $\mathbf{P}^3$), say A_1 with the WLP and A_2 without the WLP, such that the Betti numbers of A_2 are obtained from those of A_1 by performing cancellations. We will also show that, even in an instance when all level algebras having a given Hilbert function H do not enjoy the WLP (their existence is proven by Boij and the second author in $[BZ]$, Theorem 3.2), we can obtain (of course non-level) algebras having the same Hilbert

function H and enjoying the WLP by adding suitable redundant terms to a level minimal free resolution (MFR).

Let us now state the main definitions and results we will need in this paper.

Definition-Remark 1. Let n and i be positive integers. The *i-binomial expansion of n* is

$$n_{(i)} = \binom{n_i}{i} + \binom{n_{i-1}}{i-1} + \cdots + \binom{n_j}{j},$$

where $n_i > n_{i-1} > ... > n_j \geq j \geq 1$. Under these hypotheses, the *i*-binomial expansion of n is unique (e.g., see $[BH]$, Lemma 4.2.6).

Furthermore, define

$$(n_{(i)})_{-1}^{-1} = \binom{n_i - 1}{i-1} + \binom{n_{i-1}-1}{i-1-1} + \cdots + \binom{n_j - 1}{j-1},$$

and

$$n_{<i>} = \binom{n_i - 1}{i} + \binom{n_{i-1}-1}{i-1} + \cdots + \binom{n_j - 1}{j},$$

where we set $\binom{m}{q} = 0$ whenever $m < q$.

Theorem 2 (Green). *Let h_d be the entry of degree d of the Hilbert function of R/I and let L be a generic linear form of R. Then the degree d entry h_d' of the Hilbert function of $R/(I,L)$ satisfies the inequality*

$$h_d' \leq (h_d)_{<d>}.$$

Proof. See $[Gr]$, Theorem 1. $\square$

The following is a simple - but fundamental - observation of Stanley (see $[St]$, bottom of p. 67, where this remark appears in a more general form):

Lemma 3 (Stanley). *Let R/I be an artinian algebra, and let $L \in R$ be a linear*

form not belonging to I. Then the Hilbert function H of R/I can be written as

$$H: \ h_0 = c_0 = 1, h_1 = b_1 + c_1, ..., h_e = b_e + c_e, h_{e+1} = 0,$$

where

$$B: \ b_1 = 1, b_2, ..., b_e, b_{e+1} = 0$$

is the Hilbert function of $R/(I:L)$ (with the indices shifted by 1 to the left), and

$$C: \ c_0 = 1, c_1, ..., c_e, c_{e+1} = 0$$

is the Hilbert function of $R/(I, L)$.

Proof. This decomposition of H is an immediate consequence of the fact that

$$0 \longrightarrow R/(I:L)(-1) \xrightarrow{\cdot L} R/I \xrightarrow{\cdot 1} R/(I, L) \longrightarrow 0$$

is an exact complex. $\square$

Finally, we recall a theorem of Wiebe, which will be fundamental for us here:

Theorem 4 (Wiebe). *Let A be a Gotzmann algebra with Hilbert function*
$H: \ 1, h_1, h_2..., h_e, h_{e+1} = 0$, and let t be the smallest integer such that $h_t \le t$. Then A
enjoys the WLP if and only if, for all $i = 1, 2, ..., t - 1$, we have

$$h_{i-1} = ((h_i)_{(i)})_{-1}^{-1}.$$

Proof. See $[Wi]$, Theorem 4.2. $\square$

The formula for H provided by Wiebe's characterization basically means that h_{t-1} can be arbitrary (of course subject to Macaulay's theorem) but that, by Bigatti-Geramita's $[BG]$, Lemma 3.3, any other h_i, for $i \le t - 2$, is uniquely determined as the minimal possible entry of degree i of H, given h_{i+1}. Equivalently, the lex-segment (and hence every) algebra having Hilbert function H cannot have socle in degrees lower than $t-1$. However,

conversely, at each step the growth from degree i to degree $i+1$ is not necessarily maximal (e.g., $h_2 = 6$ grows maximally to 10 in degree 3, but $((h_3)_{(3)})^{-1}_{-1} = 6$ for all $h_3 = 8, 9, 10$).

We are now ready to prove our main result, namely the characterization of the finite Hilbert functions which force *all* of their (artinian) algebras to enjoy the WLP. Strangely enough, they turn out to be same which force only their *Gotzmann* (artinian) algebras to enjoy the WLP!

Theorem 5. *Let H: $1, h_1, h_2..., h_e, h_{e+1} = 0$ be a possible Hilbert function (according to Macaulay's theorem), and let t be the smallest integer such that $h_t \leq t$. Then all the artinian algebras having Hilbert function H enjoy the WLP if and only if, for all $i = 1, 2, ..., t - 1$, we have*

$$h_{i-1} = ((h_i)_{(i)})^{-1}_{-1}.$$

Proof. The "only if" part immediately follows from Wiebe's Theorem 4 (if all algebras with Hilbert function H satisfy the WLP, then in particular the Gotzmann algebras do, and hence H has the desired form).

Hence assume that $h_{i-1} = ((h_i)_{(i)})^{-1}_{-1}$ for $0 < i < t$. We want to prove that all artinian algebras $A = R/I$ with Hilbert function H enjoy the WLP. Since, clearly, H increases up to degree $t - 1$ and $h_{t-1} \geq h_t$ (by definition of t and by construction of H), then by the definition of Weak Lefschetz Property it is easy to see that A has the WLP if and only if, for any generic linear form $L \in R$, the Hilbert function of $R/(I, L)$ is

$$1, h_1' = h_1 - 1, h_2' = h_2 - h_1, ..., h_{t-1}' = h_{t-1} - h_{t-2}, h_t' = 0. \tag{1}$$

By Green's Theorem 2, since $h_t \leq t$, we clearly have

$$h_t' \leq (h_t)_{<t>} = \left(\binom{t}{t} + \binom{t-1}{t-1} + ... \binom{t-h_t+1}{t-h_t+1} \right)_{<t>} = 0,$$

whence $h_t' = 0$, as we wanted to show.

Thus, from now on, let $0 < i < t$. It is a trivial exercise, by the Pascal triangle

inequality, to show that $h_i - (h_i)_{<i>} = ((h_i)_{(i)})_{-1}^{-1}$. Hence, by Green's theorem, we have that

$$h_i' \leq (h_i)_{<i>} = h_i - ((h_i)_{(i)})_{-1}^{-1} = h_i - h_{i-1}. \tag{2}$$

On the other hand, since $I \subset I : L$, by Lemma 3 we must have that $h_i - h_i' \leq h_{i-1}$, i.e. that

$$h_i' \geq h_i - h_{i-1}. \tag{3}$$

Thus, from (2) and (3), we get (1), and the proof of the theorem is complete. □

As we noticed before, by Wiebe's theorem, our Theorem 5 can be rephrased in the following surprising form:

Corollary 6. *Let H be a finite Hilbert function. Then all algebras having Hilbert function H enjoy the WLP if and only if* any Gotzmann *algebra having Hilbert function H enjoys the WLP.*

A very interesting consequence to our Theorem 5 is the following result (that, as we said in the introduction, was first proven in $[HMNW]$, Proposition 4.4):

Corollary 7 ($[HMNW]$). *All codimension 2 artinian algebras enjoy the WLP.*

Proof. It is immediate to see, by the growth condition imposed by Macaulay's theorem on Hilbert functions H starting with 1,2,..., that they must all have the form $H : h_0 = 1, h_1 = 2, ..., h_{t-1} = t, h_t \leq t, ...$

Since, clearly, $((i+1)_{(i)})_{-1}^{-1} = i$, the result immediately follows from Theorem 5. □

Also notice that, from our Theorem 5, one implication of Wiebe's Theorem 4 immediately follows (of course, not the same that we used in our proof!):

Corollary 8 (Wiebe). *Let H: $1, h_1, h_2..., h_e, h_{e+1} = 0$ be the Hilbert function of an*

artinian algebra such that, for all $i = 1, 2, ..., t-1$, $h_{i-1} = ((h_i)_{(i)})_{-1}^{-1}$. Then all Gotzmann ideals with Hilbert function H satisfy the WLP.

Since the Betti numbers of lex-segment ideals - and therefore, more generally, of Gotzmann ideals - are the largest possible for all the algebras with a given Hilbert function (see $[Bi]$, $[Hu]$ and $[Pa]$), at this point it is natural to wonder if our theorem is the consequence of a more general fact: that is, is it true that, given an artinian algebra A_1 with the Weak Lefschetz Property, then all the algebras A_2, with the same Hilbert function as A_1 but Betti numbers strictly lower than those of A_1, enjoy the WLP as well?

Unfortunately (or not?), the answer is negative, as we will show in the next proposition. Indeed, we can even provide counterexamples which are constructed as artinian reductions of one-dimensional algebras of points of $\mathbf{P}^3$, where both A_1 and A_2 are level algebras.

Also, we will show that even in cases (whose existence was proven in $[BZ]$, Theorem 3.2) where a Hilbert function H forces all of its level algebras to not enjoy the WLP, one can find (of course non-level) algebras, with the same Hilbert function H and Betti numbers obtained by adding redundant terms to a level MFR, which enjoy the WLP.

Notice that, when A_1 and A_2 have exactly the same Betti numbers, the fact that the answer to the above question is negative is already known, even in the Gorenstein case - see $[HMNW]$, Example 3.10, where a codimension 4 Gorenstein algebra A_2 with the WLP is provided, with A_2 having the same Betti numbers and the same Hilbert function 1,4,10,10,4,1,0 of the well-known non-WLP example of Ikeda ($[Ik]$, Example 4.4).

Proposition 9. *i). There exist reduced sets of points Y_1, Y_2 in $\mathbf{P}^3$ with the same Hilbert function, and both having* level *coordinate ring, such that their (arbitrary) artinian reductions, A_1 and A_2 respectively, are such that A_1 has the WLP while A_2 does not, and the graded Betti numbers of A_2 can be obtained from those of A_1 by cancellation of redundant terms.*

ii). There exists a finite Hilbert function which forces all of its level algebras to fail to

have the WLP, but such that one of its non-level algebras has Betti numbers larger than those of a level algebra and enjoys the WLP.

Proof. i). Our artinian level reductions A_1 and A_2, quotients of $R = k[x, y, z]$, will both have Hilbert function

$$1, 3, 6, 10, 12, 12, 0. \tag{4}$$

We begin with A_1. In this case it is somewhat more intuitive to first construct the artinian algebra and then lift to points Y_1. The idea is based on the fact that (4) coincides with the Hilbert function of 12 points in $\mathbf{P}^2$ through degree 5, and such an ideal has projective dimension 1.

Let Z be a k-configuration of type (1,2,4,5); Z consists of 12 points in $\mathbf{P}^2$ with Hilbert function

$$1, 3, 6, 10, 12, 12, 12, \ldots. \tag{5}$$

Specifically, Z consists of a union of five points on a line l_1, four points on a line l_2, two points on a line l_3 and one additional point. By $[GHS]$, Z has extremal Betti numbers among sets of points in $\mathbf{P}^2$ with Hilbert function (4). The Betti diagram for R/I_Z is

```
1   -   -
-   -   -
-   -   -
-   3   2
-   2   2
```

Now we consider the ideal $I = I_Z + (x, y, z)^6$. Clearly R/I has Hilbert function (4), and the following level Betti diagram is forced from the Hilbert function and the above Betti diagram:

$$
\begin{array}{cccc}
1 & - & - & - \\
- & - & - & - \\
- & - & - & - \\
- & 3 & 2 & - \\
- & 2 & 2 & - \\
- & 12 & 24 & 12
\end{array}
\tag{6}
$$

Finally, $A_1 = R/I$ clearly has the WLP since up to and including degree 5 it coincides with the coordinate ring of a set of points in $\mathbf{P}^2$, which has depth 1. To construct the set of points, Y_1, described in the statement of the proposition, let C be a corresponding arithmetically Cohen-Macaulay union of plane curves of total degree 12 (for instance take C to be a cone over Z). By $[GMR]$, there exists a set, Y_1, of 44 points on C whose Hilbert function is the desired truncation.

We now construct the level algebra A_2 using the methods of $[Mi]$. We will find a set of points Y_2 in $\mathbf{P}^3$ directly, as the union $Y_2 = X_1 \cup X_2$, based on the computation (in the notation of $[Mi]$):

deg	0	1	2	3	4	5
Δh_{X_1}		1	3	6	7	6
Δh_{X_2}	1	2	3	4	5	6
Δh_{Y_2}	1	3	6	10	12	12

(7)

To achieve this, let X be a general set of 4 points in $\mathbf{P}^3$ and let W be a complete intersection of type $(3,3,3)$ containing X. Letting X_1 be the residual to X in W, clearly X_1 has h-vector given by the first line of (7). The MFR of I_{X_1} is easily seen to be

$$
0 \to R(-7)^6 \to R(-6)^{11} \to \begin{array}{c} R(-3)^3 \\ \oplus \\ R(-5)^3 \end{array} \to I_{X_1} \to 0.
\tag{8}
$$

It is not hard to see that one can construct an arithmetically Cohen-Macaulay curve C of degree 21, with h-vector given by the second line of (7), that contains a complete

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intersection of type (3,3,3). Indeed, one can construct C having as a component a complete intersection C_1 of type (3,3), and cut C_1 by a cubic hypersurface. Let X_2 be a general hyperplane section of C. Then X_2 also has h-vector given by the second line of (7), and we take $Y_2 = X_1 \cup X_2$. This is a basic double G-link, in the language of $[KMMNP]$, and the homogeneous ideal of Y_2 has the form

$$I_{Y_2} = L \cdot I_{X_1} + I_C,$$

where L is the linear form defining the hyperplane that cuts X_2. In particular, Y_2 has h-vector (4), as desired.

Since I_C clearly has MFR

$$0 \to R(-7)^6 \to R(-6)^7 \to I_C \to 0,$$

we use a mapping cone over the short exact sequence

$$0 \to I_C(-1) \to I_{X_1}(-1) \oplus I_C \to I_{Y_2} \to 0 \tag{9}$$

and the MFR's (8) and (9) to obtain the following level Betti diagram for R/I_{Y_2}:

$$
\begin{array}{ccccc}
1 & - & - & - \\
- & - & - & - \\
- & - & - & - \\
- & 3 & - & - \\
- & - & - & - \\
- & 10 & 24 & 12 \\
\end{array}
\tag{10}
$$

Notice that there is no possible cancellation to worry about in the mapping cone. Also, there are clearly no redundant terms. It is not hard to show, using $[Mi]$, Proposition 2.3, that any artinian reduction of Y_2 fails to have the WLP. (The key point is that for a general line λ in $\mathbf{P}^3$, the point of intersection of λ with H_4 fails to impose a condition on the linear system of quintics containing Y_2.) Finally, note that the Betti numbers of (10) can be obtained from those of (6) by cancellation (and that this involves only the first

two free modules of the MFR, since the level property is preserved). This proves point i).

ii). We will consider the case $e = 9$ of $[BZ]$, Theorem 3.2, where the Hilbert functions provided are such that all of their level algebras do not enjoy the WLP. This means that we are considering the Hilbert function

$$1, 3, 5, 7, 9, 11, 11, 8, 5, 2, 0. \tag{11}$$

As above, let $R = k[x, y, z]$. The Hilbert function (11) satisfies the condition of $[HMNW]$ for algebras having the WLP, and the construction there gives an algebra with the WLP, and having maximal Betti numbers *among algebras with the WLP* having the given Hilbert function. In this case one can check (with $[CoCoA]$) that the algebra R/I_1 constructed has MFR

$$
0 \to
\begin{matrix}
R(-9)^3 \\
\oplus \\
R(-10)^3 \\
\oplus \\
R(-11)^3 \\
\oplus \\
R(-12)^2
\end{matrix}
\to
\begin{matrix}
R(-7)^2 \\
\oplus \\
R(-8)^6 \\
\oplus \\
R(-9)^6 \\
\oplus \\
R(-10)^6 \\
\oplus \\
R(-11)^4
\end{matrix}
\to
\begin{matrix}
R(-2) \\
\oplus \\
R(-6)^2 \\
\oplus \\
R(-7)^3 \\
\oplus \\
R(-8)^3 \\
\oplus \\
R(-9)^3 \\
\oplus \\
R(-10)^2
\end{matrix}
\to R \to R/I_1 \to 0. \tag{12}
$$

On the other hand, the proof of $[BZ]$, Theorem 3.2 provides a level algebra, R/I_2, that

can be checked to have MFR

$$
\begin{array}{ccc}
 & R(-7)^2 & \\
 & \oplus & R(-2) \\
 & R(-8)^3 & \oplus \\
0 \to R(-12)^2 \to & \oplus & \to R(-6)^2 \to R \to R/I_2 \to 0 \\
 & R(-10) & \oplus \\
 & \oplus & R(-7)^3 \\
 & R(-11) &
\end{array}
$$

This algebra clearly has Betti numbers (much) below the Betti numbers of R/I_1, as we wanted to show. $\square$

Remark 10. i). By Wiebe's theorem, it is immediate to see that a Gotzmann algebra with Hilbert function (11) does not enjoy the WLP. Therefore, in the proof of Proposition 9, we have actually shown that there exist algebras failing to have the WLP whose Betti numbers are above those of an algebra, R/I_1, which has the WLP, as well as other algebras, again without the WLP, whose Betti numbers are below those of R/I_1.

ii). Notice that, actually, for *all* the Hilbert functions H provided by $[BZ]$, Theorem 3.2, which force all of its level algebras to not have the WLP, the same argument given in the proof of point ii) of Proposition 9 (employing the condition for algebras having the WLP given in $[HMNW]$) can be used to show the existence of non-level algebras with Hilbert function H enjoying the WLP.

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